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FACTORIAL EXPERIMENTS

by

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Notes (Part 3) on a series of 8 seminars on Principles of Experimental Design

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FACTORIAL EXPERIMENTS

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Factorial experiments are suitable in:

1) exploratory work where the object is to determine quickly the effects of several factors at various levels.

2) investigating the interactions among the effects of several factors.

3) investigations directed towards finding that combination of the levels of factors which will produce a maximum response.

Two factors are said to be independent if the presence of one has no effect on the other. In this case, the simple effects of a factor are the same as the main effects. For instance, if a chemical reaction involves two factors: time and temperature, each at two different levels, say 5.5 hrs, 6.5 hrs and 260°, 340° respectively, then we consider the following tables of responses of the experiment:

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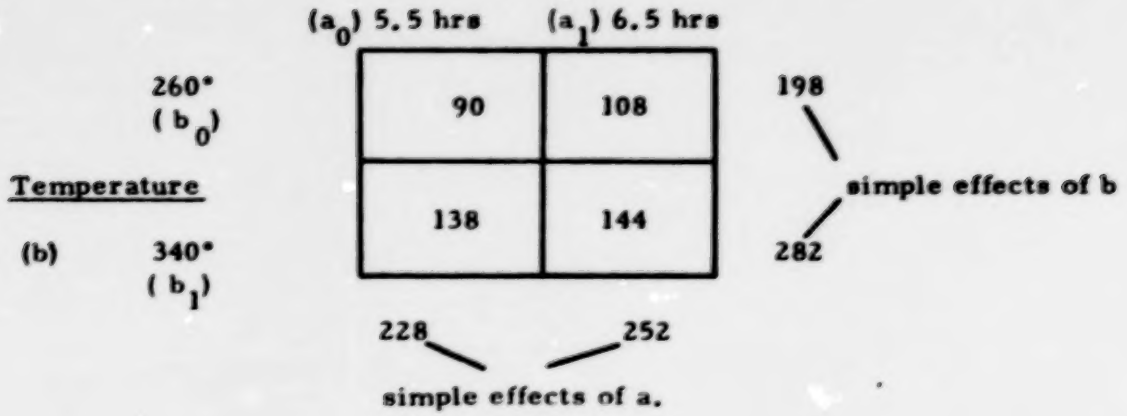

Main effect of factor a : 252 - 228 = 24

Main effect of factor b : 282 - 198 = 84

Interaction effect : (144-138) - (108-90) = - 12.

If a has no influence on b, then the interaction effect must be zero.

Suppose that in the above example the 4 different treatments viz: a_0b_0, a_0b_1, a_1b_0, a_1b_1, are replicated 6 times and the yield given by the following table:

	1	2	3	4	5	6	Total
a_0b_0	16	22	16	10	18	8	90
a_0b_1	28	27	17	20	23	23	138
a_1b_0	16	25	16	16	19	16	108
a_1b_1	28	30	19	18	24	25	144
							480

Analysis for Contrasts

	a_0b_0	a_0b_1	a_1b_0	a_1b_1	Effect	SS contrasts
a_1 vs a_0	- 1	- 1	1	1	24	24
b_1 vs b_0	- 1	1	- 1	1	84	294
Interaction	1	- 1	- 1	1	- 12	6
						324

The SS for contrasts are obtainable thus:

a_1 vs a_1 : $$\frac{24 \times 24}{\left[(-1)^2 + (-1)^2 + 1^2 + 1^2\right] \times 6} = 24$$

b_1 vs b_0 : $$\frac{84^2}{\left[(-1)^2 + 1^2 + (-1)^2 + 1^2\right] \times 6} = 294$$

Interaction: $$\frac{12^2}{\left[1^2 + (-1)^2 + (-1)^2 + 1^2\right] \times 6} = 6$$

Trt. SS : $$\frac{90^2 + \cdots\cdots + 144^2}{6} - \frac{480^2}{24} = 324$$

'a' SS : $$\frac{228^2}{12} + \frac{252^2}{12} - \frac{480^2}{24} = 24$$

'b' SS : $$\frac{198^2}{12} + \frac{282^2}{12} - \frac{480^2}{24} = 294$$

Interaction SS : $\frac{(90 + 144)^2}{12} + \frac{(138 + 108)^2}{12} - \frac{480^2}{24} = 6$

Total SS : $16^2 + \ldots\ldots\ldots + 25^2 - \frac{480^2}{24}$

A. O. V.

Source	df.	SS	MS	F	
Total	23	724			
Trts	3	324	108	13.5	
due to a	1	24	24	3.0	
" " b	1	294	294	36.75*	significant
" " ab	1	6	6	0.75	
Replicates	5	280	56		
Error	15	120	8		

At the 5% level $F_{5,15} = 2.9$

$F_{3,15} = 3.29$

$F_{1,15} = 4.54$

Thus it is clear that all variations are due to the main effect of factor b. The effects of a and the interaction appear to be negligible.

The analysis of multiple factors at higher levels follows the same pattern. For details see Cochran and Cox. The techniques used in the exploration of response surfaces, and the method of steepest ascent are well explained in Cochran and Cox in Chapter 8A.

SAMPLE SIZE PROBLEMS

Chebychev's lemma: If a random variable X with mean value μ and variance σ^2 is considered in an experiment, one of the problems associated with it is the determination of the probability with which X lies between $(\mu - k\sigma)$ and $(\mu + k\sigma)$ where k is a prescribed constant. Chebychev's lemma states that:

$$P\left\{\mu - k\sigma < X < \mu + k\sigma\right\} > 1 - \frac{1}{k^2} \qquad \text{Eq. 1}$$

Using this result, rough bounds on the sample size needed to attain prescribed confidence with a given probability can be determined.

In Eq. 1 if we use $\bar{x}$ for x, then σ is changed into $\sigma/\sqrt{n}$ where n is the sample size. Thus Eq. 1 becomes

$$P\left\{\mu - k\frac{\sigma}{\sqrt{n}} \leq \bar{x} < \mu + k\frac{\sigma}{\sqrt{n}}\right\} > 1 - \frac{1}{k^2}$$

If the maximum permissible error between $\bar{x}$ and μ given to be d then

$$d = k\,\sigma/\sqrt{n} \qquad \text{Eq. 2}$$

Given d, k and σ, n can be evaluated from Eq. 2.

Example 1. $d = \frac{1}{10}$, $1 - \frac{1}{k^2} = \frac{9}{10}$, $\sigma^2 = 0.15$.

Since $k^2 = 10$, from Eq. 2

$$n = \frac{k^2\sigma^2}{d^2} = \frac{10\sigma^2}{(1/100)} = 1000\,\sigma^2 = 150.$$

Example 2. How many times must a balanced coin be tossed so that the observed number of heads departs from the true mean by not more than 0.1 with a probability of 0.9?

For a balanced coin $\mu = \frac{1}{2}$, $\sigma^2 = \frac{1}{4}$,

$$\frac{k\sigma}{\sqrt{n}} = 10, \quad k^2 = 10$$

Therefore $n = 100k^2\sigma^2$

$= 1000\,\sigma^2 = 250$

For more examples on the sample size problem vide Cochran and Cox, Chapter 2.

REFERENCE

Cochran and Cox: Experimental Designs, 2nd Ed. John Wiley and Sons, New York, 1957.